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FIXED POINT THEOREMS FOR (α, ψ)-MEIR-KEERLER-KHAN MULTIVALUED MAPPINGS IN b-METRIC SPACE

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Abstract

The aim of this paper is to establish fixed points for multivalued mappings,by adapting the ideas in [1] to the b-metric space setting.

Key words: (α, ψ)-Meir-Keerler-Khan multivalued mappings;b-metric space;α-admissible function;fixed point.

1 Introduction

The notion of metric space,introduced by Féchet in 1906,is one of the cornerstone of not only mathematics but also several quantitative sciences.Due to its importance and application potential,this notion has been extended,improved and generalized in many different ways.An incomplete list of the results of

such an attempt is the following :quasi-metric space ,symmetric space,fuzzy metric space and so on.
In this paper,we pay attention to the concept of b-metric space.The notion of b-metric space was introduced by Czerwik[19] in 1993 to extend the notion of metric space.In this interesting paper,Czerwik [19] observed a characterization of the celebrated Banach fixed point theorem [20] in the context of complete b-metric spaces.Following this pioneer paper,several authors have devoted their attention to research the properties of a b-metric space and have reported the existence and uniqueness of fixed points of various operators in the setting of b-metric spaces(see,e.g.,[10,21-30] and some reference therein).
The aim of this paper is to generalize various known results proved by Zhigang Wang and Huilai Li[1] to the case of b-metric spaces and give an example to illustrate our main results.

2 Preliminaries

Let $\mathbb{R}^+$ denote the set of all nonnegative real numbers and let $\mathbb{N}$ denote the set of positive integers.From [19,23,31,32]we get some basic definitions,lemmas,and notations concerning the b-metric space.

Definition 2.1. *Let X be a nonempty set and let $s \geq 1$ be a given real number.A function $d\colon X \times X \to \mathbb{R}^+$ is said to be a b-metric if and only if for all $x, y, z \in X$ the following conditions are satisfied :*
(1)$d(x, y) = 0$ if and only if $x = y$;
(2)$d(x, y) = d(y, x)$;
(3)$d(x, z) \leq s[d(x, y) + d(y, z)]$.
Then,the triplet (X, d, s) is called a b-metric space.

It is an obvious fact that a metric space is also a b-metric space with $s = 1$,but the converse is not generally true.To support this fact,we have the following example.

Example 2.1. *Consider the set $X = [0, 1]$ endowed with the function $d\colon X \times X \to \mathbb{R}^+$ defined by $d(x, y) = |x - y|^2$ for all $x, y \in X$.Clearly,$(X, d, 2)$ is a b-metric space but it is not a metric space.*

Let (X, d, s) be a b-metric space.The following notations are natural deductions from their metric counterparts.
(i)A sequence $\{x_n\} \subseteq X$ converges to $x \in X$ if $\lim_{n\to\infty} d(x_n, x) = 0$.
(ii) A sequence $\{x_n\} \subseteq X$ is said to be a Cauchy sequence if,for every

given $\epsilon > 0$,there exists $n(\epsilon) \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$ for all $m, n \leq n(\epsilon)$.
(iii) A b-metric space (X, d, s) is said to be complete if and only if each Cauchy sequence converges to some $x \in X$.
From the literature on b-metric spaces,we choose the following significant example.

Example 2.2. *(see[33]) Let $p \in (0,1)$.Consider the space $L^p([0,1])$ of all real functions $f : [0,1] \to \mathbb{R}$ such that $\int_0^1 |f(t)|^p dt < +\infty$,endowed with the functional $d : L^p([0,1]) \times L^p([0,1]) \to \mathbb{R}$ defined by*
$d(f, g) = (\int_0^1 |f(t) - g(t)|^p dt)^{\frac{1}{p}}$ $\forall$ $f, g \in L^p([0,1])$.
Then,$(X, d, 2^{\frac{1}{p}})$ is a b-metric space.

Next,we collect some lemmas and notions concerning the theory of multivalued mappings on b-metric spaces.We recall that $CB(X)$ denotes the class of nonempty closed and bounded subsets of X.For $A, B \in CB(X)$,define the function $H : CB(X) \times CB(X) \to \mathbb{R}^+$ by
$H(A, B) = max\{\delta(A, B), \delta(B, A)\}$,
where

$$\delta(A, B) = \sup\{d(a, B), a \in A\}, \qquad \delta(B, A) = \sup\{d(b, A), b \in B\} \tag{1}$$

with $d(a, C) = \inf\{d(a, x), x \in C\}$
Note that H is called the Hausdorff b-metric induced by the b-metric d.
We recall the following properties from [19,22,23];see also [34] and the references therein.

Lemma 2.3. *Let (X, d, s)be a b-metric space.For any $A, B, C \in CB(X)$ and any $x, y \in X$,one has the following: (i)$d(x, B) \leq d(x, b)$,for any $b \in B$;*
(ii) $\delta(A, B) \leq H(A, B)$;
(iii) $d(x, B) \leq H(A, B)$,for any $x \in A$;
(iv) $H(A, A) = 0$;
(v)$H(A, B) = H(B, A)$;
(vi)$H(A, C) \leq s(H(A, B) + H(B, C))$;
(vii)$d(x, A) \leq s(d(x, y) + d(y, A))$.

Remark 2.4. *The function $H : CL(X) \times CL(X) \to \mathbb{R}^+$ is a generalized Hausdorff b-metric; that is,$H(A, B) = +\infty$ if $\max\{\delta(A, B), \delta(B, A)\}$ does not exist.*

Lemma 2.5. *Let (X, d, s) be a b-metric space.For $A \in CL(X)$and $x \in X$,one has*
$d(x, A) = 0 \Leftrightarrow x \in \bar{A} = A$,
where $\bar{A}$ denotes the closure of the set A.

Lemma 2.6. *Let (X, d, s) be a b-metric space and $A, B \in CL(X)$.Then,for each $h > 1$ and for each $a \in A$ there exists $b(a) \in B$ such that $d(a, b(a)) < hH(A, B)$if $H(A, B) > 0$.*

Finally,to prove our results we need the following class of functions. Let $s \geq 1$ be a real number ;we denote by Ψ_s the family of strictly increasing functions $\psi\colon [0, +\infty) \to [0, +\infty)$ such that
$\sum_{n=1}^{+\infty} s^n \psi^n(t) < +\infty$ for each $t > 0$,
where ψ^n denotes nth iterate of the function ψ.It is well known that $\psi(t) < t$ for all $t > 0$.An example of function $\psi \in \Psi_s$ is given by $\psi(t) = \frac{ct}{s}$ for all $t \geq 0$,where $c \in (0, 1)$.

Definition 2.2. *A multivalued mappings $T\colon X \to CL(X)$ is said to be α-admissible,with respect to a function $\alpha\colon X \times X \to [0, \infty)$,for each $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq 1$,we have $\alpha(y, z) \geq 1$ for all $z \in Ty$.*

Definition 2.3. *Let (X, d, s) be a b-metric space and let $\delta(., .)$ be as in (4).Then ,a multivalued mappings $F\colon X \to CL(X)$ is said to be h-upper semicontinuous at $x_0 \in X$,if the function*
$\delta(Fx, Fx_0) := \sup\{d(y, Fx_0) : y \in Fx\}$
is continuous at x_0.Clearly,F is said to be h-upper semicontinuous,whenever F is h-upper semicontinuous at every $x_0 \in X$.

Definition 2.4. [1] *Let $T\colon X \to K(X)$ be a mapping on a metric space (X, d).Then T is called an (α,ψ)-Meir-Keeler-Khan multivalued mapping if there exist $\psi \in \Omega$ and $\alpha\colon X \times X \to [0, \infty)$ such that*

$$H(Tx, Ty) \neq 0 \Rightarrow \alpha(x, y)H(Tx, Ty) \leq \psi(P(x, y)) \tag{2}$$

for any $x, y \in X$,where

$$P(x, y) = \frac{d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)}{d(x, Ty) + d(y, Tx)}.$$

Theorem 2.7. [1] *Let $T\colon X \to K(X)$ be an (α, ψ)-Meir-Keerler-Khan multivalued mapping on a metric space (X, d, s).Suppose that the following hypotheses hold:*
(i) (X, d) is a complete metric space;
(ii) T is an α-admissible multivalued mapping;
(iii) There exist x_0 and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;
(iv) T is continuous.
Then there exists a fixed point of T in X.

Definition 2.5. *Let $T\colon X \to CB(X)$ be a mapping on a metric space (X,d,s).Then T is called an (α,ψ)-Meir-Keeler-Khan multivalued mapping if there exist $\psi \in \Psi_s$ and $\alpha\colon X \times X \to [0,\infty)$ such that*

$$H(Tx,Ty) \neq 0 \Rightarrow \alpha(x,y)H(Tx,Ty) \leq \psi(P(x,y)) \tag{3}$$

for any $x,y \in X$,where

$$P(x,y) = \frac{d(x,Tx)d(x,Ty) + d(y,Ty)d(y,Tx)}{d(x,Ty) + d(y,Tx)}.$$

3 Main results

Theorem 3.1. *Let $T\colon X \to CB(X)$ be an (α,ψ)-Meir-Keerler-Khan multivalued mapping on a metric space (X,d,s).Suppose that the following hypotheses hold:*
(i) (X,d) is a complete metric space;
(ii) T is an α-admissible multivalued mapping;
(iii) There exist x_0 and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$;
(iv) T is h-upper semicontinuous.
Then there exists a fixed point of T in X.

Proof. We construct a sequence starting from x_0.If $x_0 \in Tx_0$,then x_0 is a fixed point.Suppose $x_0 \notin Tx_0$.Because Tx_0 is a compact subset of X,then $d(x_0,Tx_0) > 0$.If $x_1 \in Tx_1$,then x_1 is a fixed point,and subsequently,this proof is complete.Assume that $x_1 \notin Tx_1$.Then it is clear that $d(x_1,Tx_1) > 0$ because Tx_1 is a compact subset of X.We have

$$\begin{aligned} H(Tx_0,Tx_1) &\leq \alpha(x_0,x_1)H(Tx_0,Tx_1) \\ &\leq \psi(\frac{d(x_0,Tx_0)d(x_0,Tx_1) + d(x_1,Tx_1)d(x_1,Tx_0)}{d(x_0,Tx_1) + d(x_1,Tx_0)}) \\ &= \psi(d(x_0,Tx_0)) \end{aligned}$$

Moreover,by the definition of the Hausdorff metric and the fact that $x_1 \in Tx_0$ we get

$$d(x_1,Tx_1) \leq H(Tx_0,Tx_1) \leq \psi(d(x_0,Tx_0)). \tag{4}$$

In addition,the compactness of Tx_1 implies that there exists $x_2 \in Tx_1$ such that

$$d(x_1,x_2) = d(x_1,Tx_1). \tag{5}$$

In combination with equations (4)and (5),we obtain that

$$d(x_1, x_2) \leq \psi d(x_0, Tx_0) \tag{6}$$

We continue the sequence similarly.If $x_2 \in Tx_2$,then this proof is done.Thus,we assume that $x_2 \notin Tx_2$.Because $\alpha(x_0, x_1) \geq 1$ and $x_1 \in Tx_0$,$x_2 \in Tx_1$,we have $\alpha(x_1, x_2) \geq 1$.
Furthermore,using condition(3) we obtain that

$$\begin{aligned} H(Tx_1, Tx_2) &\leq \alpha(x_1, x_2) H(Tx_1, Tx_2) \\ &\leq \psi\left(\frac{d(x_1, Tx_1)d(x_1, Tx_2) + d(x_2, Tx_2)d(x_2, Tx_1)}{d(x_1, Tx_2) + d(x_2, Tx_1)}\right) \\ &= \psi(d(x_1, Tx_1)) \end{aligned} \tag{7}$$

and,subsequently,

$$\begin{aligned} d(x_2, Tx_2) &\leq H(Tx_1, Tx_2) \\ &\leq \psi(d(x_1, Tx_1)) \\ &= \psi(d(x_1, x_2)). \end{aligned} \tag{8}$$

Likewise,by the compactness of Tx_2,there exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) = d(x_2, Tx_2). \tag{9}$$

In combination with equations (6),(8)and(9),we obtain that

$$\begin{aligned} d(x_2, x_3) &\leq \psi(d(x_1, Tx_1)) \\ &= \psi(d(x_1, x_2) \\ &\leq \psi^2(d(x_0, Tx_0)). \end{aligned} \tag{10}$$

By induction,we can obtain a sequence $\{x_n\}$ satisfying

$$x_{n+1} \in Tx_n, \qquad x_{n+1} \notin Tx_{n+1}, \qquad \alpha(x_n, x_{n+1}) \geq 1$$

and

$$d(x_n, x_{n+1}) \leq \psi^n(d(x_0, Tx_0)) \tag{11}$$

for all $n \in \mathbb{N}$.
Let $m > n$.Then

$$\begin{aligned}
d(x_n, x_m) &\leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \\
&\leq \sum_{k=n}^{m-1} s^k \psi^k(d(x_0, Tx_0)), \\
&\leq \sum_{k=n}^{m-1} s^k \psi^k(d(x_0, x_1)),
\end{aligned}$$

and so $\{x_n\}$ is a Cauchy sequence in X.Hence,there exists $z \in X$ such that $x_n \to z$.
From

$$\begin{aligned}
d(z, Tz) &\leq s[d(z, x_{n+1} + d(x_{n+1}, Tz))] \\
&\leq sd(z, x_{n+1}) + s\delta(Tx_n, Tz),
\end{aligned}$$

since T is h-upper semicontinuous,passing to limit as $n \to +\infty$,we get

$$d(z, Tz) \leq 0 \tag{12}$$

which implies $d(z, Tz) = 0$.Finally,since Tz is closed we obtain that $z \in Tz$;that is,z is a fixed point of T. □

Example 3.2. *Let $X = R$,and let $d(x, y) = |x - y|$ for all $x, y \in X$ Define a mapping $T\colon X \to CB(X)$ by*

$$T(x) = \begin{cases} \{0\} & (x = 0) \\ \{\frac{5}{6}x\} & (0 < x \leq 1) \\ \{\frac{4}{x}\} & (x > 1). \end{cases} \tag{13}$$

Let

$$\psi(t) = \begin{cases} \{\frac{6}{7}t\} & (t \geq 1) \\ \{\frac{5}{6}t\} & (0 \leq t < 1). \end{cases} \tag{14}$$

Then,$\psi \in \Psi_s$ and ψ is a strictly increasing function.
Let $\alpha\colon X \times X \to [0, \infty)$ be defined by

$$\alpha(x, y) = \begin{cases} 6 & (0 \leq x, y \leq 1) \\ 0 & \textit{otherwise}. \end{cases} \tag{15}$$

Obviously, condition (iii) of Theorem 3.1 is satisfied with $x_0 = \frac{1}{6}$.
We show that (3) is satisfied .
Let $x, y \in X$ *be such that* $\alpha(x, y) \geq 1$.
Then,$0 \leq x, y \leq 1$.
If $x = y$,*then obviously (3)is satisfied .*
Let $x \neq y$.
If $x = 0$ *and* $0 < y \leq 1$,*then we obtain*

$$\begin{aligned} H(Tx, Ty) &= H(0, \frac{5}{6}y) \\ &\leq \frac{5}{6} \leq \psi(d(x, Tx)) \leq \psi(P(x, y)). \end{aligned}$$

Let $0 < x \leq 1$ *and* $0 < y \leq 1$.
Then,we have

$$\begin{aligned} H(Tx, Ty) = d(Tx, Ty) &= d(\frac{5}{6}x, \frac{5}{6}y) \\ &= \frac{5}{6}|x - y| = \psi(d(x, y)) \\ &\leq \psi(P(x, y)). \end{aligned}$$

Thus,(3) is satisfied.
We now show that T *is* α*admissible.*
Let $x \in X$ *be given,and let* $y \in Tx$ *be such that* $\alpha(x, y) \geq 1$.
Then,$0 \leq x, y \leq 1$.
Obviously,$\alpha(y, z) \geq 1$ *for all* $z \in Ty$ *whenever* $0 < y \leq 1$.
If $y = 0$,*then* $Ty = \{0\}$.*Hence,for all* $z \in Ty$,$\alpha(y, z) \geq 1$.
Hence,T *is* α*-admissible.Thus,all hypotheses of Theorem 3.1 are satisfied.However,0 and 2 are the two fixed points of* T.

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